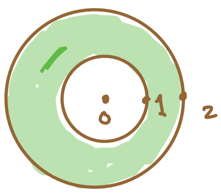
Math 4200
Monday November 2
3.3 Laurent series. We'll prove the Laurent series theorem for analytic functions in annuli, which we illustrated on Friday with rational function examples.

Announcements:

- Jess, Carlie, Austin : zeta fcn project

warmup: Laurent series for $\frac{1}{z-2} + \frac{1}{z+1}$ in annulus $1 < |z| < 2$?



big

$$\frac{1}{-2}\,\frac{1}{1-z/2}$$

$$-\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}z^n$$

$\left|\frac{z}{2}\right| < 1$

$|z| < 2$

converges inside big disk

$$\frac{1}{z}\,\frac{1}{1-\left(-\frac{1}{z}\right)}$$

$$\frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n}$$

$\left|\frac{1}{z}\right| < 1$

$|z| > \frac{1}{1} = 1$

converges outside little disk

<u>Laurent Series Theorem</u> For $0 \le R_1 < R_2$ let

- $A = \{ z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2 \}$

be an open annulus (or punctured disk in case $R_1 = 0$). Then (1) and (2) below are equivalent, and the uniqueness of Laurent coefficients (3) also holds:

(1) $f : A \to \mathbb{C}$ is analytic.

(2) $f(z)$ has a power series expansion using non-negative and negative powers of $(z - z_0)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} \frac{a_{-m}}{(z - z_0)^m}.$$

$$:= S_1(z) + S_2(z).$$

$2 \Rightarrow 1$ because uniform limits of analytic fns are analytic.

- Here $S_1(z)$ converges for $|z - z_0| < R_2$ and uniformly absolutely for
- $|z - z_0| \le r_2 < R_2$. • And $S_2(z)$ converges for $|z - z_0| > R_1$, and uniformly absolutely for $|z - z_0| \ge r_1 > R_1$.

- (3) The Laurent coefficients a_k, $k \in \mathbb{Z}$ are uniquely determined by f. Specifically, if γ is any p.w. C^1 contour in A, with $I(\gamma, z_0) = 1$, e.g. any circle of radius r, with $R_1 < r < R_2$, then

$2 \Rightarrow 3$.

"Residue"

$$a_{-1} = \frac{1}{2\pi i} \int_\gamma f(\zeta)\, d\zeta \quad : \quad \int_\gamma f(\zeta)\, d\zeta = \int_\gamma \sum_{n=0}^{\infty} a_n (\zeta - z_0)^n \, d\zeta + \int_\gamma \sum_{m=1}^{\infty} \frac{a_{-m}}{(\zeta - z_0)^m} \, d\zeta$$

by unif conv.

$$= \sum_{n=0}^{\infty} \int_\gamma a_n (\zeta - z_0)^n d\zeta + \sum_{m=1}^{\infty} \int_\gamma \frac{a_{-m}}{(\zeta - z_0)^m} d\zeta$$

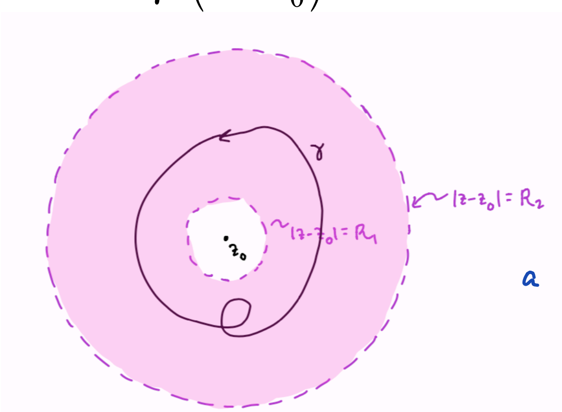
$(\zeta - z_0)^k$ has antideriv $k \in \mathbb{Z}$ unless $k = -1$

So FTC $\Rightarrow$ all zero except $\frac{1}{\zeta - z_0}$ one

$$= a_{-1} \int_\gamma \frac{1}{\zeta - z_0} d\zeta = a_{-1} 2\pi i\, I(\gamma; z_0) = (a_{-1})(2\pi i)$$

and more generally, each a_k is given by

- $$a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta \quad (2)$$



(2) Consider $g(\zeta) = \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}$ also analytic in annulus.

a L.S. for $g(\zeta)$ is $\frac{1}{(\zeta - z_0)^{k+1}}$(L.S. for f)

$\frac{1}{2\pi i} \int_\gamma g(\zeta) d\zeta$ = Res for g. coeff of $\frac{1}{\zeta - z_0}$ term for g, which was the coeff of $(\zeta - z_0)^k$ for f was a_k.

We'll discuss $(2) \Rightarrow (1)$, $(2) \Rightarrow (3)$ on this page, and $(1) \Rightarrow (2)$ on the next page.

proof of (1) $\Rightarrow$ (2) in the Laurent series theorem:

<u>Laurent Series Theorem</u> For $0 \le R_1 < R_2$ let
$$A = \{ z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2 \}$$
be an open annulus (or punctured disk in case $R_1 = 0$). Then (1) and (2) below are equivalent, and the uniqueness of Laurent coefficients (3) also holds:

(1) $f: A \to \mathbb{C}$ is analytic.

(2) $f(z)$ has a power series expansion using non-negative and negative powers of $(z - z_0)$:
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} \frac{a_{-m}}{(z - z_0)^m}.$$
$$:= S_1(z) + S_2(z).$$
Here $S_1(z)$ converges uniformly absolutely for any compact subdisk $|z - z_0| \le r_2 < R_2$.
And $S_2(z)$ converges uniformly absolutely for $|z - z_0| \ge r_1 > R_1$.

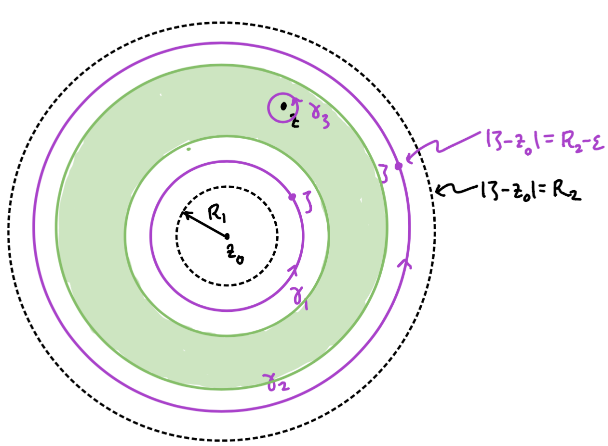
proof: Let $\varepsilon > 0$. Consider z in the compact subannulus $R_1 + 2\varepsilon \le |z - z_0| \le R_2 - 2\varepsilon$. Let γ_1 be the circle $|\zeta - z_0| = R_1 + \varepsilon$, let γ_2 be the circle $|\zeta - z_0| = R_2 - \varepsilon$, and let γ_3 be a concentric circle arround z or radius less than ε. (See figure.) Use the section 2.2 replacement theorem for the first equation below, and then the CIF for the second one:

green region

$$\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{\gamma_3} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad = 2\pi i\, f(z) \quad \text{C.I.F.}$$

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta$$

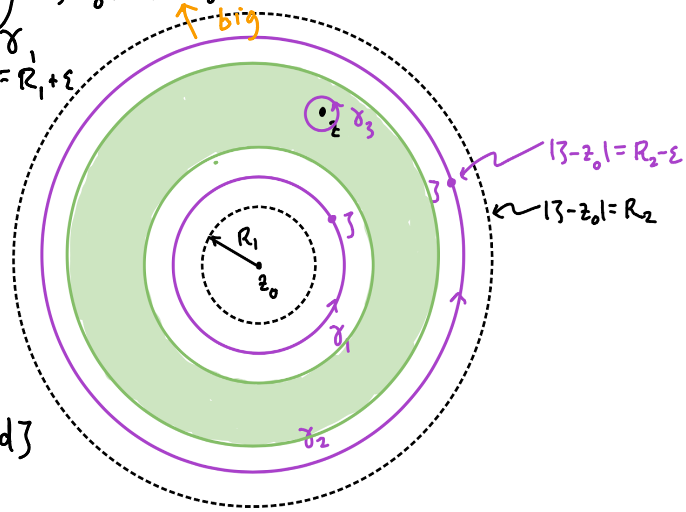
Now use our geometric series wizardry to find the Laurent expansion for $f(z)$!



$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\underbrace{(\zeta - z_0)}_{\text{big}} - (z - z_0)}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{(\zeta - z_0) - \underbrace{(z - z_0)}_{\text{big}}}\,d\zeta$$

$|\zeta - z_0| = R_2 - \varepsilon$ $\qquad$ $|\zeta - z_0| = R_1 + \varepsilon$



$$= \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z_0}\,\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}\,d\zeta$$

- $\left|\frac{z - z_0}{\zeta - z_0}\right| \le \frac{R_2 - 2\varepsilon}{R_2 - \varepsilon} < 1$ on γ_2

$$+ \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{(z - z_0)}\,\frac{1}{1 - \frac{\zeta - z_0}{z - z_0}}\,d\zeta$$

- $\left|\frac{\zeta - z_0}{z - z_0}\right| \le \frac{R_1 + \varepsilon}{R_1 + 2\varepsilon} < 1.$

$$= \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z_0}\sum_{n=0}^{\infty}\frac{(z - z_0)^n}{(\zeta - z_0)^n}\,d\zeta + \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{z - z_0}\sum_{k=0}^{\infty}\frac{(\zeta - z_0)^k}{(z - z_0)^k}\,d\zeta$$

uniform conv to interchange ints with sums
also factor powers of $(z - z_0)$

$$= \sum_{n=0}^{\infty}(z - z_0)^n \underbrace{\left(\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta\right)}_{a_n} + \sum_{k=0}^{\infty}\frac{1}{(z - z_0)^{k+1}}\,\frac{1}{2\pi i}\int_{\gamma_1} f(\zeta)(\zeta - z_0)^k\,d\zeta$$

$(m = k+1)$ $\qquad$ $a_{-m} = \frac{1}{2\pi i}\int_{\gamma_1} f(\zeta)(\zeta - z_0)^{m-1}\,d\zeta$

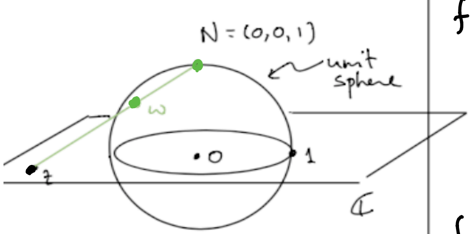
On the next two pages we use Laurent series to classify *isolated singularities*, and give equivalent geometric conditions which characterize the three kinds: removable singularities, poles, essential singularities. And we will revert to the text's lettering for the coefficients of the positive and negative powers in a Laurent series. $f(z)$ has an *isolated singularity at* z_0 means that there is some radius $r > 0$ so that f is analytic in the punctured disk $D(z_0, r) \setminus \{z_0\}$ We write the Laurent series as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} \frac{\overset{(a_{-m})}{b_m}}{(z - z_0)^m}$$

Isolated singularities table

Let f analytic in $D(z_0; r) \setminus \{z_0\}$, some $r > 0$

type of singularity @ z_0	definition: Laurent series definition	geometric characterization (behavior of f near z_0)
removable (because f extends to be analytic @ z_0	• $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ $\Longrightarrow$ (no negative powers in L.S.)	any of: ① $\lim_{z \to z_0} f(z) = L \in \mathbb{C}$ exists $\Downarrow$ ② $\lvert f(z) \rvert \le M \ \forall\ 0 < \lvert z - z_0 \rvert \le \rho$ some $0 < \rho < r$ $\Downarrow$ ③ $\lim_{z \to z_0} f(z)(z-z_0) = 0$
pole (North pole!) of order $N \in \mathbb{N}$ • simple pole if $N = 1$	$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=1}^{N} \frac{b_m}{(z-z_0)^m}$ with $b_N \neq 0$ $f(z) = \frac{1}{(z-z_0)^N}\left[b_N + b_{N-1}(z-z_0) + \dots + b_1 (z-z_0)^{N-1} + a_0 (z-z_0)^N + \dots\right]$ $f(z) = \frac{1}{(z-z_0)^N} g(z)$, g analytic @ z_0 $\lim_{z \to z_0} f(z) = \infty$	① $\lim_{z \to z_0} f(z) = \infty$ (the north pole of the Riemann sphere) • ② $\exists\, N \in \mathbb{N}$ s.t. $g(z) = (z-z_0)^N f(z)$ has a removable singularity at $z = z_0$, with $g(z_0) \neq 0$



$z = x + iy$

$w = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2} \right)$

$x + iy = \frac{w_1}{1-w_3} + i \frac{w_2}{1-w_3} \quad (w_3 \neq 0)$

• $\lvert z \rvert \to \infty$ iff $w \to (0,0,1)$

type of singularity @ z_0	Laurent series def.	geometric characterization
essential singularity	$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z-z_0)^m}$ • with $b_{m_j} \neq 0$ for some seq. $\{m_j\} \to \infty$.	$\forall\ 0<\rho<r$, $\overline{f(D(z_0;\rho) \setminus \{z_0\})} = \mathbb{C}$! (In fact, more is true and is called "Picard's Thm": $f(D(z_0;\rho)\setminus\{z_0\})$ contains all of $\mathbb{C}$ except for at most a single point! $\forall\ 0<\rho<r$) e.g. $f(z) = e^{\frac{1}{z}}$ @ $z_0 = 0$ $f(D(0;\rho)\setminus\{0\}) = \mathbb{C}\setminus\{0\}$ $\forall \rho > 0$.

\# to be cont'd